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AUTHOR Souviney, Randall J.
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ABSTRACT

This paper, part of a series of working papers published to provide documentation of research and development efforts associated with the Indigenous Mathematics Project of Papua New Guinea, discusses the role of problem solving in the mathematics curriculum and types and characteristics of good problems, with illustrations. Reasoning by analysis and synthesis are presented in some detail, with attention to ends-means analysis and structured problem solving. In the next section, the search for problem solutions is discussed, including suggestions for organizing problem solving sessions. Polya's four-step plan is described, with comments on each step. Several problem-solving strategies are illustrated, with a sample problem worked for each: guesstimation, work backwards, reduce to a simpler case, conduct an experiment, and add elements to the problem situation. In the final section, classroom activities are presented, such as whole-class problem-solving warm-ups and three problems with solutions. A bibliography is included. (MNS)

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INDIGENOUS MATHEMATICS PROJECT

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STRATEGIES FOR MATHEMATICAL PROBLEM SOLVING

Randall J. Souviney

Working Paper 2
March 1980

THE INDIGENOUS MATHEMATICS PROJECT

This series of working papers and research reports has been published to provide a comprehensive source of documentation on research and development efforts associated with the Indigenous Mathematics Project. The Project was established in 1977 by the Government of Papua New Guinea to investigate various aspects of traditional mathematics development. During the first phase (1977–79) of the five year program, basic research in crosscultural cognitive development was carried out and indigenous counting, classification and measurement systems used throughout the country were documented. During the second phase of the project (jointly funded by the Department of Education and UNESCO), pilot instructional materials were developed and trialled. The intent was to assess the feasibility of utilizing complementary aspects of indigenous and Western mathematics as a basis for developing culturally relevant student materials, instructional aids and teachers' guides which reflect the practical constraints of the community school environment. The results of IMP work are intended to inform future curriculum development in an effort to provide more appropriate instructional materials and learning aids for Community Schools throughout Papua New Guinea.

Series Editor: Randall Souviney
Managing Editor: Jacquie Wathen

STRATEGIES FOR MATHEMATICAL PROBLEM SOLVING

Randall Souviney
UNESCO Advisor

Indigenous Mathematics Project
March 1980

This paper was developed based on sections of an early draft of a book by the author, Solving Problems Kids Care About, Santa Monica, California: Goodyear Publishing Company, 1981. It was originally presented in an abbreviated form at an Education Research Unit seminar at the University of Papua New Guinea, March 1980.

What's Worth Knowing

As our world changes before our eyes--our knowledge base becoming obsolete at an alarming rate...value systems which seem to change with or without us--what do we teach our children who must survive and prosper under conditions of a changing environment we may scarcely be able to imagine? What is worth teaching today that will be worth knowing in the year 2000?

Two educational issues come to mind which hold the greatest promise for helping children become independent, competent actors in their world. First, in virtually every field accepted practices are becoming obsolete as rapidly as new knowledge develops. For example, in teaching it becomes more and more difficult to remain current as the years pass. Teachers must search for the essential strands which transcend the blizzard of competing ideologies and techniques in order to function effectively in their classrooms. Specialization, the traditional mechanism designed to contend with information explosion, will continue to flourish. The time is long past when one individual possessed a thorough understanding of an entire field. (Karl Gauss is purported to be the last individual to understand all contemporary mathematics, and he died over one hundred years ago. We have developed more mathematics in the past twenty years than in the thousands of years before Gauss!)

Clearly, we must look at learning differently if the children in our elementary schools today are to gain a sense of competence in their lives tomorrow. Facts and relationships between facts will become less important, learned temporarily and later replaced by a new set of relevant facts and relationships. We are discovering that many of the "facts" associated with everyday life are reflections of our own values and may not be valid in other societies. Ethnocentric thinking, though comfortable,

will be less useful in our shrinking, multicultural world.

A knowledge of mathematics will be increasingly important in tomorrow's world although certain elements will be more useful than others. Computation will continue to play an important role, however we will depend more and more on the speed and accuracy of the calculator for large computations. The ability to understand and solve naturally occurring problems will be a prized attribute. While facts are important only so far as they provide a medium for understanding a particular problem, creative solution straegies, once developed, remain useful when confronting a wide range of situations. There is little value to methodically deriving the wrong answer to ten decimal places.

A second challenge to the education community which may have even greater impact on what is worth knowing is the availability of versatile low-cost personal computers. These crafty little devices are truly technological marvels, unheard of only a few years ago. The machine that sits on my desk while I'm writing this introduction costs less than $1500 and is more powerful than the computer that serviced the entire university only twelve years ago valued at several million dollars. If development continues over the next five years, a machine 1000 times more powerful will be available for under $500. Literally millions of these personal computing devices will be challenging and entertaining people by the middle 1980's.

The education community cannot afford to ignore the potential impact such devices will have on learning and curriculum design. Access to vast stores of factual information (history, literature, sports statistics, election results, geographic facts, languages, etc.) will depend on how well individuals "communicate" with their computers. We may soon see the day when you can ask your personal computer to argue with your bank's computer to resolve disputes in your checking account. The microcomputer

promises to convert our most popular spectator sport, watching television, into a highly stimulating interactive activity. The ultimate effect on our society as a whole is sure to be at least as significant as the introduction of the printing press or the onset of the industrial revolution.

What does this mean for elementary education today? Learning to compute will remain an important part of the math curriculum; however, conceptual understanding and estimation skills will take precedence over accuracy and speed with large computations. Use of ratio fractions will diminish in favor of early introduction of decimals. Computing devices (calculators, special purpose devices, microcomputers) will be utilized in increasing numbers at all levels of education. If used properly, these machines will become a vital factor in our efforts to enhance the ability of our students to solve mathematical problems in school and throughout their lives.

Problems Worth Solving

Before turning our attention to specific problem solving teaching strategies, we must first discuss the characteristics of problem situations which are appropriate for novice solvers. (Here, novice refers to level of experience which may have little to do with one's age.) If our goal is to provide classroom experiences which encourage divergent and logical thinking on the part of our students, consider the following criteria for problem selection.

An ideal problem situation should:

1. Be readily understandable to the student, yet the solution should not be immediately apparent.
2. Be intrinsically motivating and intellectually stimulating.
3. Have more than one solution "path".
4. Require only previously learned arithmetic operations.
5. Lend itself to being solved over a reasonable period of time (not a simple computational procedure).

6. Be somewhat open-ended (solutions should suggest new problems).

7. Intergrate various subject areas-- mathematics, science, social studies, fine arts.

8. Be well enough defined so you will know when it's solved.

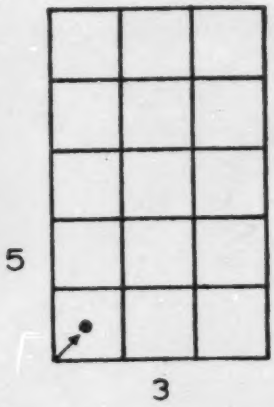
Convergent and Divergent Problems

These eight criteria do not in general characterize the type of problems found in elementary textbooks. Word problems are generally presented in a convergent manner whereby the student decides which operation(s) to apply to the value provided in order to determine the correct answer. This type of presentation offers a rather restricted view of problem solving.

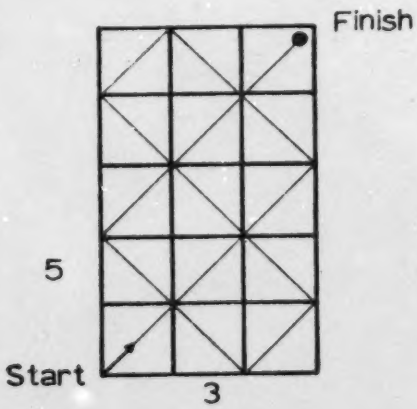
Though many problems encountered in life involve quantities (counting, measures, money), rarely are these values presented in an organized, symbolic form. An individual first must recognize that a problem exists and be motivated to seek understanding. This initial, seemingly aimless, trial-and-error stage of problem solving generally includes doodling, sketching, estimating, measuring, counting, asking questions and listening. With children it is especially important to encourage the exploration of ideas and concrete materials as the means to defining problem parameters and eventual solution strategies. Most of all,such problems offer a glimpse of the creative act which teachers, engineers, accountants, mothers, politicians, architects and scientists are engaged in daily - namely the dissecting (analysis) of complex events and the piecing together (synthesis) of familiar elements to derive order and meaning from novel problem situations. The following example is illustrative of the type of divergent problem situation which is easy to describe yet requires a bit of "messing about" with ideas and sketches in order to fully define the problem situation and predict an appropriate solution strategy.

Billiard Table Math

Suppose you have a rectangular billiard table like the one below. If you shoot out from the lower left corner at 45° (split the corner square), which corner will the ball end up in?



First, it is not at all clear that the ball will end up in any corner. Also, you have to realize that a billiard ball bounces off at the same angle it strikes the cushion (45° in this case). It might require a bit of experimenting in order to justify this <u>one</u> element of the problem. One convenient way to test out your skill is to sketch the table on 1 cm graph paper and trace the path of the ball.



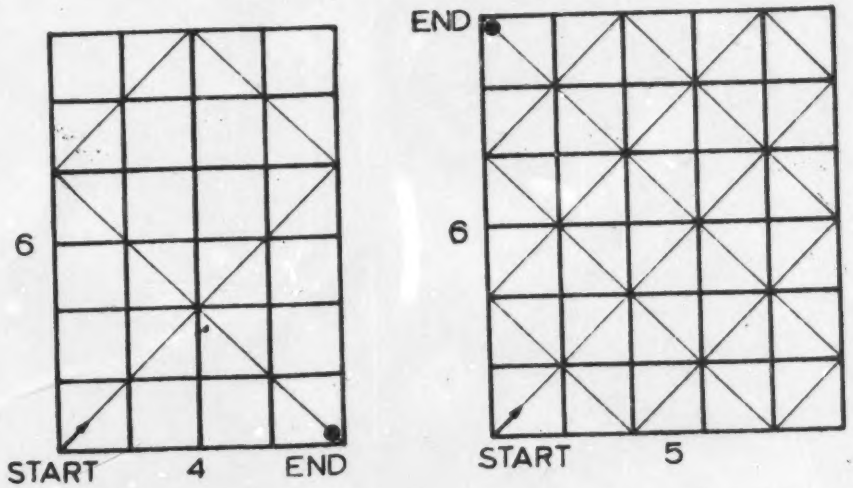
This odd sized table (conventional size is 3 x 6) provides an "interesting" game, finishing in the upper right corner. Notice the ball crosses every square. It this true for other sized rectangular tables? Will the ball always end in a corner (assume the dimensions are whole units)? Can

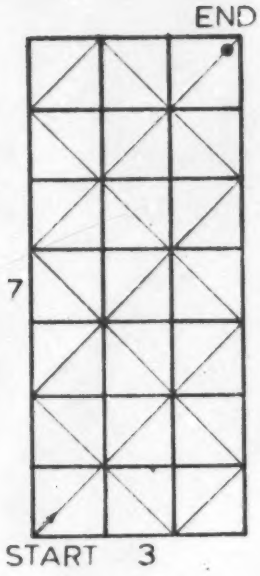
you predict which corner the ball will end in without tracing its path. (To keep things simple remember to shoot out from the lower left corner at 45°.)

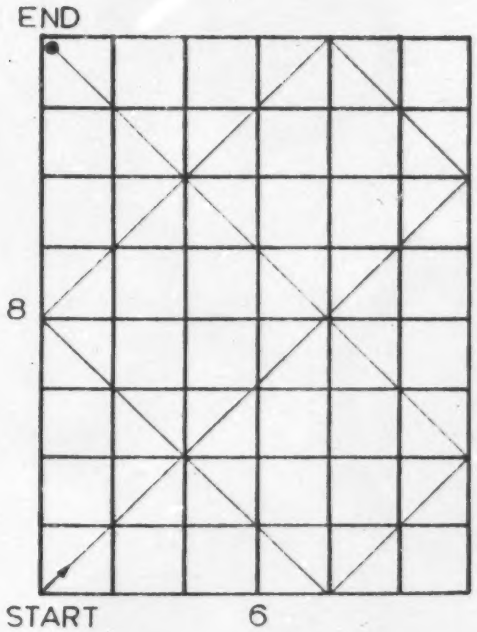
Each of these questions is easy to understand, but the solution is not immediately apparent. The solution to the initial problem raised several new interesting questions. The intellectual and motor skills necessary to effectively participate in this problem should be available to most children nine or older. The situation is motivating to most kids and intellectually stimulating. It's reasonably easy to determine when you have solved the problem. Based on our problem selection criteria, this problem seems to be an ideal candidate for classroom use.

Let's continue our game. One strategy is to try out tables of various sizes and see if any useful insight is gained from the results or if a pattern emerges. This state of experimentation not only allows the collection of potentially useful data, but also encourages a high level of involvement on the part of the solver. As we will discuss later, motivation and mind set play a significant role in effective problem solving.

"Interesting" or "Boring" Game







Do table dimensions determine if a game is <u>interesting</u> (every square crossed by the ball) or <u>boring</u> (some squares not crossed)? Perhaps a table would help organize the data we collected. Using graph paper, check out the data in the table below. Play a few additional games and enter that data as well. Note that the first number of each pair represents the length of the base of the table (horizontal axis) and the second represents the height (vertical axis).

Billiard Games

Interesting	Boring
(3, 5)	(4, 6)
(5, 6)	(6, 8)
(3, 7)	(4, 10)
(4, 9)	(3, 9)
(5, 4)	(4, 4)
(,)	(,)
(,)	(,)

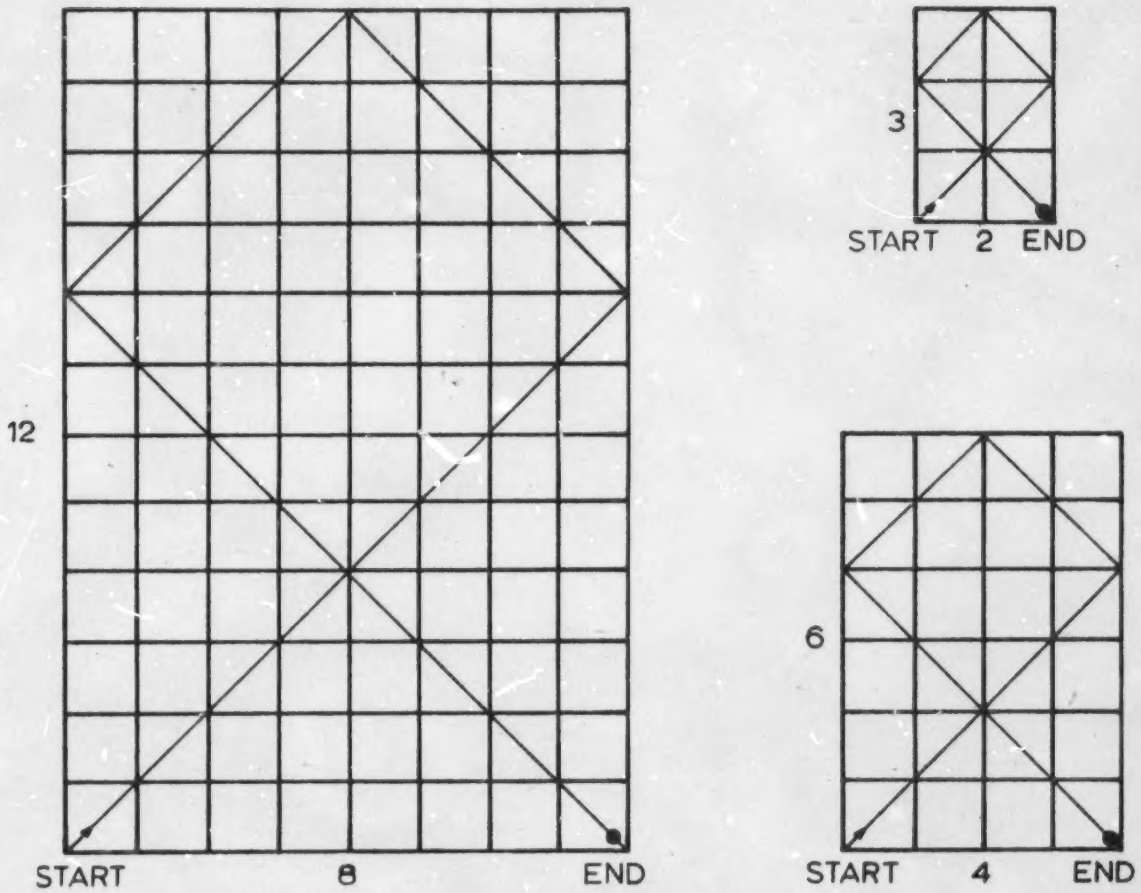
Looking at the number pairs, put your pattern recognition skills to work. How are the number pairs for the interesting games alike? How are they different from all the boring games? At this point in the solution process, past experience looking for number patterns and recent experience in generating examples of billiard games must somehow foster the critical moment of inspiration. There are no guarantees in creative problem solving. If the light doesn't shine one has three choices:

1. Continue generating data using the present strategy, hoping some key element is just around the corner;
2. Try a new strategy;
3. Let the problem "rest" a few days and then try 1 or 2 above.

The inspiration is this case involves observing the relationship between the first and second number in each pair. Notice that for an interesting game, the number pairs have no factors (other than 1) in common [e.g., (3, 7), (3, 5), (5, 6)]. This is not the case for boring games

[e.g., (4, 6), (4, 10), (6, 8)]. Another way of looking at (modeling) the situation is to consider each pair a fraction or ratio. Interesting games can't be reduced (e.g., 3/7, 3/5, 5/6) while boring games can (e.g., 4/6, 4/10, 6/8). A "family" of billiard tables can be constructed opening a whole new area of investigation.

Family of Games



Notice the similar pattern the ball makes for each game in the (2, 3) family. Is this true for other families [e.g., (3, 4), (6, 8), (9, 12)]. Will the ball end in the same corner for every game in a family? Will this information help us predict which corner the ball will end up in

for all games? Completing a table like the one below and collecting data on interesting games only, should make the situation more transparent. (If you have done your homework, you should have found that every boring game has an interesting family member that ends in the same corner.)

The Ball Stops Here

Lower Left	Upper Left	Upper Right	Lower Right
	(3, 4)	(3, 7)	(4, 5)
	(5, 6)	(5, 9)	(6, 7)
	(7, 8)	(5, 7)	(4, 9)
	(,)	(,)	(,)

Again, recognizing a pattern that distinguishes between the columns should inspire an "Ah-ha" experience. A simple way to describe the solution is that, for an interesting game, the ball always stops at the end of the even axis. If there is no even axis [e.g., (5, 7)], the ball ends in the upper right hand corner. (It never returns to its starting point.)

The last step in the process is to convince yourself that your solution is true for all cases. Generally, it is not necessary to construct a formal proof at this level. Unless it is a life and death situation or you're risking the family fortune, working out a few well chosen examples should suffice. This "proof by desire" should be held tentatively, ready to be altered as necessary in order to accommodate any conflicting data that might arise in subsequent experiments.

Word Problems

Though word problems have been accused of fostering everything from math anxiety to premature baldness, when used wisely they provide a convenient means for practicing certain problem solving skills. A few simple procedures can change those end-of-the-chapter demons into effective learning activities.

First of all, word problems should not be used as cleverly disguised drill activities. They should instead be used to provide much needed experience in deciphering problem situations under controlled conditions.

To give children practice translating English into the language of mathematics, give children "headlines" and have them write stories which fill in the details.

$2 \times 12 = 24$ (headline)

Students in Ms. Miller's class decided to take pictures of all the teachers in the school. They earned enough money for two rolls of film with 12 pictures on each roll. They can take 24 pictures in all.

Experiences with headlines help children convert mathematical sentences into realistic dialogue. The opposite procedure is also useful. Have students write an appropriate headline for problems found in their texts. It is not necessary to compute solutions in order to write headlines. Avoiding the actual computations allows children to concentrate on deciphering the problem situation and devising a solution plan. A calculator can be useful at this stage as well. For example, a possible headline for the problem below might be: "$12 \div 4 = 3$ omelets."

> Problem: If 4 people decide to share equally a dozen eggs for breakfast how many will each person get?

Textbook word problems are carefully chosen to be convergent --that is, require a limited number of steps which lead to one "correct" answer. Though such problems are convenient, the solution strategies they require generally involve recognizing when to apply an appropriate rule--the most difficult strategy for beginners. Any analysis of the problem is generally

quite limited while the synthesis of appropriate computations comprises most of the solution effort.

Problems which succumb to conducting an experiment, organizing data or observing a pattern should become part of your classroom's daily mathematical diet. You will find many problems of this type included in the following sections as well as in the references listed in the bibliography. Using word problem activities such as Headlines provides practice with English to mathematics translations. However they should be mixed with more divergent problems to encourage the development of analytical problem solving skills. Remember, problems encountered in the real world are almost never presented in written form. It is through the process of defining and understanding the problem situation and its conditions that the seeds of solution are sown.

A Word About "Real World" Problems

Problems found in math books generally differ from real world problems in two important ways. The success you and your students enjoy when applying the techniques discussed here and in the next chapter to practical problems encountered in daily life will depend greatly on your ability to understand and compensate for these distinctions.

First, some problems in everyday life may have no solution under the naturally occurring conditions. Remember those carefully stated conditions found in most "school" problems: you can remove one marble at a time, return it to the bag and shake. The implicit conditions are also relatively transparent: don't look up the answer in the back of the book! With real world problems the conditions, both stated and implicit, become far more important.

For example, consider the impending world oil crises. Prior to 1979, most people realized that petroleum supplies were limited and were

rapidly being depleted by an energy-hungry world. The problem for the United States became one of meeting the growing energy demand of industry and the public. Several "solutions" were suggested including conservation, increasing efforts to find new domestic oil fields, synthetic fuels, higher import quotas and alternative energy sources. The easiest solution, increasing import quotas, is no longer working. The problem remains the same but the conditions have changed. The complex Mideast situation has seriously affected sales of crude. Atomic power is under fire. Long and troubled waits in gas lines have become commonplace. Many people can not afford to heat their homes during the winter.

Real world problems are rarely solved permanently. When trouble arises, new solutions must be found to meet new conditions. Strategies originally considered and discarded may now take on new promise. For example, synthetic fuels made from plentiful coal and shale are suddenly a viable alternative with gasoline selling at over $1.00 a gallon. Success with such problems is as much due to an ability to adapt to explicit conditions as it is to understanding the processes of analysis and synthesis.

The second distinction between "school" problems and "folk" problems involves the underlying value structure of the solver. If the solution to a problem is persistently elusive, always check your assumptions. Though this advice is generally sound, when a values conflict is suspected it is absolutely essential.

A values conflict occurs when two individuals experience the same event and arrive at two opposing points of view and, based on their own value system, both are absolutely correct. When this occurs, a solution to the conflict may be impossible unless one or both sets of values are compromised. Values conflicts occur between age groups, political parties, socio-economic groups and especially in multicultural environments.

These practical social problems which many times show up in classrooms, customer-relations desks at department stores and in the international political arena are often exasperatingly difficult to solve. One common pitfall is failing to recognize that a values conflict is at the heart of the problem in the first place--the solver assumes everyone involved in the situation has a common understanding of the events. Solutions to "real world" problems often require a deep understanding of each other's value system, considerable mutual respect and an overall balance of power. Violating these conditions often invites a micro-war in the classroom or a real war if sufficient national interest is at stake.

The techniques described in this book are useful in solving an interesting, though necessarity limited class of problems. It is the author's contention that in the real world such methods certainly contribute to the solution of practical problems but are not sufficient, in themselves, to unravel the complexities of human interaction.

Two Types Of Reasoning

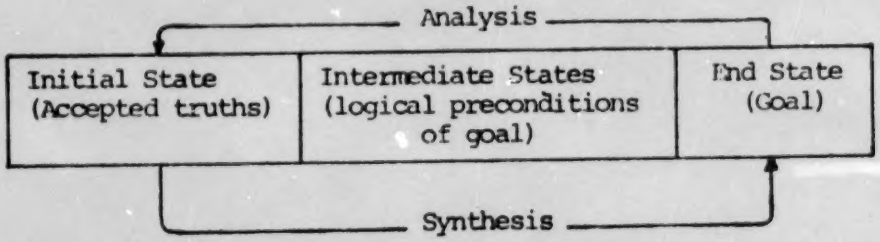
Regressive and progressive reasoning are two characteristic viwes of creative problem solving (the solution of novel problems). Both were originally described by the Greeks and have been subsequently refined over the past two thousand years by countless mathematicians and scientists. Analysis and synthesis have complementary features which, when applied appropriately, aid greatly in planning and implementing a solution strategy (see figure 1).

Analysis (regressive reasoning) is essentially the intellectual task of logically working through a problem backwards. Beginning with a well defined (though presently obscure) problem situation, the solver searches for a simpler condition which, if true, will guarantee the solution of the original situation. For example, if you found yourself stranded miles from the nearest gas station because a prankster let the air out of your bicycle tire, a simple precondition which guarantees you will be back on the road is to somehow refill your tire. By continuing the process of

deducing appropriate preconditions, you eventually arrive at a condition that can be easily satisfied--perhaps turning the bicycle upside down and temporarily filling the tire with water. Your analysis should lead directly to a plan which can then be carried out by repeating the steps in the opposite direction (simplest to most complex) The complementary process is called synthesis (progressive reasoning). Any inadvertent error in logic made during the analytical phase will be graphically discovered during the implementation of the solution plan---it may take all night to fill the tire with water!

Most problems require the coordinated application of both processes though as problems become more routine, less analysis may be needed.

Insert figure 1 about here



Analysis

Analysis is the process of simplifying a currently obscure situation by initially assuming the goal is true and applying known logical operations to establish a set of logical preconditions subsequently arriving at a known or transparent state. The primary distinction between analysis and synthesis is that with analysis all the information necessary to solve the problem situation is embodied in the goal statement though it may be initially obscure and unintelligible. Analysis continually reduces the complexity of the situation until the interrelationships become clear. Carefully describing the problem goal is vital if a successful solution is to be found. Chess players, lawyers, doctors and detectives rely heavily on analysis for their problem solving needs.

Synthesis

Synthesis is the process of applying known processes (from a wide range of possible processes) to given factual information (initial state) in order to attain a specific goal (end state). As the initial state is adapted by each process, the relationship between the current state and the desired goal is evaluated and subsequent processes applied in order to eventually reduce the misfit between the goal and the current state to zero. Many unique solution paths may be possible for such problems. Learning to ride a bicycle or build a model airplane are examples of synthetic problems.

Figure 1

Ends-Means Analysis

An alternative way of thinking about these reciprocal methods of reasoning involves a process called ends-means analysis. The ends refer to a clear statement of the goal or intent. The better the "ends" are described for a given problem, the easier the solution. The means consist of a set of known concepts and principles which can be brought to bear on the given situation.

The first step when applying ends-means analysis is to state your goal clearly so you will know when you have solved the problem. In many cases, this initial step will require a good deal of thought and experimentation. The results of your efforts, however, will be well worth the trouble.

Once a clear goal has been established, the next step is to determine a sequence of actions which, if carried out, will insure a satisfactory solution (analysis). Finally, as each step in the solution process is carried out (synthesis), the condition of the current state of the situation is examined and compared to the desired result. If the degree of misfit diminishes, the plan is continued. If not, the problem and solution plan is re-examined. Ends-means analysis embodies the analysis of the problem situation into a sequence of actions along with a scheme for monitoring the syntheses of these actions working towards a solution.

Regardless of what convenient structure we utilize to explain creative problem solving, an intangible element remains which defies rigorous description yet seems fundamental to the entire enterprise. Factual knowledge and past problem solving experience are certainly important; however, an element of chance pervades any novel problem situation. An effective problem solver must not only possess appropriate skills and knowledge to bring to bear on the situation, but must have the motivation, persistence and self-confidence to risk failure in pursuing a satisfactory solution.

Though chance favors the prepared mind, it takes courage to see a truly novel problem through to a satisfactory conclusion.

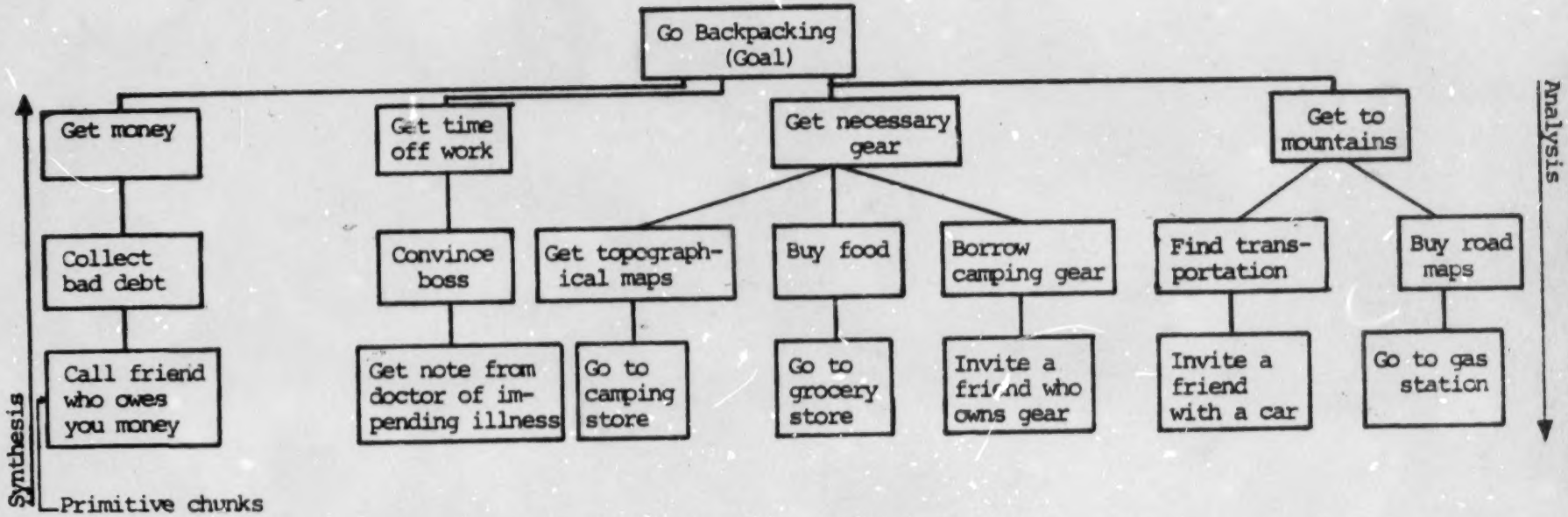
Structured Problem Solving

An additional way to view the problem solving process is borrowed from the field of computer science. Structured problem solving involves first establishing a well defined goal. Using analysis, a "tree" of subgoals is determined which are preconditions for the desired result. This "chunking" process is continued with each subgoal until all the individual tasks are well known (primitive). Finally, through synthesis, the process is reversed; individual subgoals are accomplished and coordinated with each other. If the process is successful, the original need will be satisfied. If not, understanding of the goal must be reviewed in light of the error and analysis resumed anew.

As the low-level chunks and their uses become more familiar to the user, less analysis may be required. When a set of related chunks become completely generalized, a whole class of problems can be addressed by relying on a synthesis of these well known chunks. For example, in attempting to design a hull which is very stable in rough seas, a novice might have to build several models before discovering the optimum combination of length, width, depth and shape (analysis). An experienced builder brings to bear a well coordinated set of skills (chunks) which allow him to confidently predict hull performance characteristics (synthesis). The tree below provides a practical example of planning a backpacking trip.

Many implicit conditions apply to the practical problem of going backpacking. Each situation is unique. The tree is but one example of the key considerations which would be taken into account when planning a trip. The primitive chunks (terminal tasks on each branch) must be coordinated in order to synthesize a solution. For example if money isn't obtained prior to going to the store, embarrassment may issue. It is

Structured Problem Solving Tree



possible to make the solution more elegant (efficient) if, for example, all the trips to the store are combined in one journey. It is interesting to note that the set of primitive (known) chunks which solve a particular problem may not be unique. Their order as well as their content may vary. An additional general problem solving plan is described in chapter 2. The original verson of this plan was described by Polya and is utilized throughout the remainder of this book as the means of solving the problems presented.

Summary

Learning effective strategies and skills for solving novel problems is becoming more important in our rapidly changing world. It is vital that schools prepare children for a world which is certain to be far more complex and challenging than today. To develop such problem solving skills it is necessary to carefully choose problem situations which offer a balance of divergent and convergent elements. A problem worth solving:

1. Should be understandable yet the solution should not be readily apparent.
2. Is motivating and easy to describe.
3. Has more than one solution path.
4. Requires skills and concepts appropriate for the grade level.
5. Should be solvable over a reasonable time period.
6. Is open ended whereby solutions suggest new questions to be investigated.
7. Should intergrate several subject areas (science, social studies, fine arts, mathematics).
8. Should be well defined so you will know when it's solved.

The more a particular problem exhibits these criteria, the greater potential it has for developing healthy problem solving attitudes and

skills in children and all novice problem solvers. In general, a "good" problem should require a high level of involvement on the part of the solver-making models or sketches, doing experiments, collecting and organizing data and searching for patterns and generalizations. Such interaction enhances the child's understanding of the problem situation, generates clues not originally offered in the problem description and assists in selecting an appropriate solution strategy.

Word problems can be used as a means for practicing various problem solving skills. Writing stories to describe a number sentence "headline" helps children translate mathematics into everyday English and visa-versa. Word problems should not be used as cleverly discuised arithmetic drill.

Real world problems differ from typical mathematics problems in two important ways. The solution of every day problems depends far more on explicit and implicit conditions then do the well defined examples found in problem books. In fact, many real world problems may have no solution at all under the existing conditions. The second distinction involves the human factor of values conflict. Many of the most difficult problems in society resist solution due to these underlying differences in world view. The strategies described in this book may be useful in addressing such problem situations but are not sufficient in themselves to unravel the complexities of human interaction.

Mathematical problem solving depends on two basic types of reasoning:

1. Analysis or regressive reasoning.
2. Synthesis or progressive reasoning.

Knowledge of these two processes is fundamental to understanding general problem solving plans such as ends-means analysis and structured problem solving. Each involves the analysis of a problem situation into a sequence of well defined actions and the synthesis of these actions in working towards a solution.

Ends-means analysis refers to a process of identifying a desired result (end) and analysing a set of actions (means) which seems to insure a satisfactory solution. Subsequently, when carrying out each step in the solution process, the desired result is compared to the current status in order to monitor progress and make appropriate adjustments in the solution.

Structured problem solving is a process of constructing a "tree" of subgoals which together, from a complete set of preconditions for the desired result. These subgoals are further subdivided until they reach a "primitive" or known state. Reversing the process completes the solution.

INVENTION

I've done it, I've done it!
Guess what I've done!
Invented a light that plugs into the sun.
The sun is bright enough,
The bulb is strong enough,
But, oh, there's only one thing wrong...

The cord ain't long enough.

Shel Silverstein
(from Where the Sidewalk Ends)

In Search Of Solutions

No single solution strategy is sufficient for all problems. Successful problem solvers may have to bring several strategies to bear on particularly novel problems before a moment of inspiration finally makes the situation transparent. The most a particular strategy can promise is to increase your understanding of the problem and its conditions enhancing the probability that some obscure relationship will become apparent. Through this initial process of investigation and discovery, a seemingly novel problem may turn out to be related to a previously solved problem and therefore subject to a similar solution.

Several general problem solving plans have been developed over the past two thousand years to assist individuals with their assault of the novel or unknown. Pappus (circa 300 A.D.) offers one of the earliest descriptions of a formal problem solving model. Most general models are relatively inaccessible to elementary aged children with one notable exception. George Polya describes a "general heuristic" in his most readable work, How to Solve It (Princeton University Press, 1973). The problem solving model he describes is easily understood and applicable to a wide range of mathematical and practical problems. This four-step plan

helps novices of all ages organize the entire solution enterprise thereby significantly increasing the probability of success in solving unique problems.

It is important to note that it is often beneficial for small groups of two to four children to work together on a common problem. Though much of schooling stresses the cognitive and affective development of the individual, practically every major scientific engineering problem solved in the past century succumbed to a team effort. It seems appropriate to begin developing these interactional problem solving skills at an early age. Brainstorming, discussion skills, data gathering, subdividing the work load, and group decision making all constitute effective group problem solving strategies.

Groups-Of-Four

One of the most effective ways of organizing problems solving sessions in your classroom is the randomly assign students into groups-of-four. Groups of this size insure that everyone has the opportunity to provide input as well as enhance the probability that enough "good" ideas are generated to keep the solution process alive.

Present your class with the problem of breaking into teams so that everyone has an equal chance of being in anyone of the groups (random assignment). Such an arrangement may be inconceivable to some children who "can't think without their friends" or "get sick when they sit next to a girl". Dealing with such real would issues is the stuff "real" understanding is made of. Teachers, who confront such situations daily, should be encouraged to view such activities as ideal opportunities for children to practice their problem solving skills in a context most meaningful to their daily classroom life. Children we have worked with have offered several suggestions for breaking into groups (counting off and drawing numbers from a hat; selecting cards from a well shuffled deck

of appropriate size) however, your class should be encouraged to develop its own procedure.

Once established, each team can be given a unique problem to work on or a set of problems from which to select. All four team members must work on the same problem. The following rule is the only one we have found necessary to enforce (beyond common sense): In order to seek help from anyone outside the group-of-four (i.e: the teacher) all four students must agree on the question to be asked.

Initial problem solving sessions should be relatively brief (10 to 15 minutes) followed immediately by a large group discussion of about equal duration. One member from each group should report their progress and ask for suggestions. With experience, longer sessions may be warranted and follow-up sessions may be less frequent, though more intense. It is important that the groups rotate the reporting responsibilities so that everyone has the opportunity to summarize results to the class.

Groups-of-four offers a flexable problem solving arrangement for students of all ages. Leaderships roles and individual responsibilities are negotiated by the children themselves. Once the groups are functioning the teacher can sit in on discussions, providing hints and asking critical questions as appropriate. The groups must work towards concensus, asking for help or reporting results as a unit. Changing group organization frequently (every 1 to 2 weeks) encourages flexibility and divergent thinking. Groups-of-four can become an integral part of your school day. Not only will it serve as a vehicle for mathematical problem solving but also may assist in resolving other critical classroom issues confronting you and your students throughout the year.

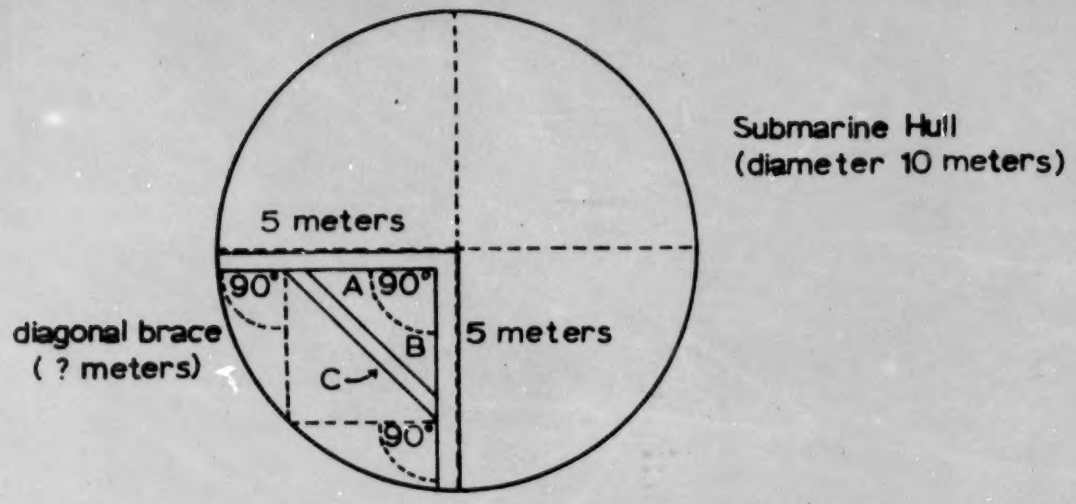
Mind Set

Several psychological factors significantly affect an individual's ability to confront novel problems successfully. As discussed previously,

self-confidence and persistence develop as a result of successful experiences with challenges of appropriate difficulty. In turn they foster an optimistic attitude that problems are essentially solvable. Without such optimism, even the most dedicated individual will be defeated before they begin.

An additional factor which seems important in developing successful problem solving skills involves mind set. Mind set refers to the level of flexibility an individual exhibits throughout the process of selecting an appropriate course of action. Early in the solution enterprise it is important to maintain an open mind when evaluating competing strategies. Many times we fall into the trap of following a particular solution path just because the scenery is familiar. The better one becomes at suspending initial judgements and actively seeking novel alternatives, the greater the likelihood of uncovering solutions to unfamiliar problems. Once a likely strategy has been determined, this open mind set gives way to a persistent search for patterns and clues. If the results are disappointing after a reasonable effort, persistence must in turn give way to flexibility in order to once again give competing strategies an opportunity to surface. It is important to maintain an open mind set throughout the first two stages in the four step plan-understanding the problem and designing a solution strategy-described in the next section. The following example suggests the importance of initially viewing problems in an open, holistic manner. For example:

> A steel fitter was told to build a shelf inside the hull of a submarine from the plans below. The diameter of the hull is known to be 10 meters. The draftsman forgot to label the length of the diagonal brace. The worker quickly figured the proper length without looking at the original blueprint.



The reader who remembers a bit of high school geometry might begin by applying the Pythagorean theorem ($A^2 + B^2 = C^2$), but will soon see their efforts as fruitless. Establishing a mind set too early in the solution process may cause us to focus on unproductive elements within the problem situation. In this case, it is useful to observe that the brace forms the diagonal of a rectangle. Knowing that the diagonals of a rectangle are equal and observing the opposite diagonal is in fact a radius of the hull, it is a simple matter to conclude that the brace is 5 meters long.

A FOUR STEP PLAN

Polya's general problem solving plan can be described as four separate yet interrelated steps:

1. Understanding the problem
2. Designing a solution strategy
3. Carrying out the strategy
4. Evaluating the results.

Step 1: Understanding the problem.

The first step in solving any problem is to understand what is given and the intended goal. This is a crucial stage in the solution process. Without a thorough understanding of what is known and where you are headed, the process of determining a solution strategy will be greatly confounded. Understanding the problem situation involves more than reading a few sentences or listening to a story. Not only are the immediate facts

important, but related facts and associated known problems as well.

Conditions relevant to the solution must also be described so they can be taken into account. The more practical (i.e., realistic) the problem, the more important are the conditions. For example, if we consider the problems associated with the excessive use of imported oil in the United States today, one simplistic solution is to legislate radically lower import quotas. This "solution" solves the stated problem, but it violates an equally important implicit condition, namely that we need to reduce imports without destroying our economy in the process. Though these implicit conditions may simply represent a minor complication in some cases, they may relate to a more general problem in others. Regardless, they must be taken into account if any solution is to be considered valid.

Some useful techniques to help children understand the facts and conditions associated with a problem include:

1. Restating the problem in their own words.
2. Listing given information.
3. Listing given conditions.
4. Writing down the stated goal in their own words.
5. Listing related relevant facts
6. Listing implicit conditions
7. Describing related known problems.

Step 2: Designing a Solution Strategy

Once the problem situation has been satisfactorily untangled, a solution strategy should be chosen from those available. This step in the plan is often applied prematurely and is a frequent cause of failure. Many times insufficient effort is spent initially with understanding the problem itself.

Strategies are chosen based on past experience with related problems, insight gathered from the structure of the given information, conditions or goal and perhaps some unexplained inspiration. The teacher can play a

vital role at this stage by offering appropriate hints to assist the discovery process. One of the greatest challenges to teachers is to know when to provide guidance and when to allow the struggle to continue unassisted. One way of viewing this subtle process is to consider that anyone can solve any problem if given sufficient expert assistance. The role of the sensitive teacher then is to broaden this zone of proximal development until the student requires little or no external assistance. Generally such knowledge applies to a narrow class of problems, though with sufficient experience it may be possible to acquire the ability to transfer skill from one domain to another. An accomplished engineer might more easily learn to be an architect than someone without a similar problem solving background.

The teacher's task becomes that of recognizing the "teaching moment" and providing an appropriate level of guidance according to the ability and experience of the learner. This ability to recognize appropriate points to interject a word of encouragement or an unobtrusive suggestion seems as rare as it is important. Questions should initially be quite general, gradually becoming more specific until the student receives just enough assistance to effectively proceed. Questions and hints should be offered discreetly and be selected in such a manner that they "could have occurred to the student himself." Avoid magic tricks and surprise results. Polya suggested several questions which can be used as models for teachers wishing to give children an appropriate share of the work in designing a solution strategy.

Helpful Hints and Questions for Problem Solving

1. What is the goal? What are you trying to find?
2. What information is given? What do you know?
3. What special conditions (restrictions) apply?
4. Try to remember a problem with a similar unknown (goal).
5. Could you use its solution to help in any way?

6. Can you restate the problem in your own words?

7. If you can't solve the problem as stated, can you first solve a similar one?

8. Did you use all the given information?

9. Are there other facts that could be useful?

10. How do the given conditions restrict the solution?

Only as a last resort should the teacher offer specific suggestions like "Maybe you should average these six amounts," or "I think I would try multiplying the number of cars by 4." Remember, learning to solve problems is like traveling-planning is half the fun .

Strategies For Problem Solving. In designing an assault on a novel problem, one or more of the following arsenal of strategies should prove to be effective. Most problems succumb to more than one technique though whole classes of problems may be successfully untangled with the judicious application of one favorite strategy.

Guesstimation: Trial and error methods are widely used by people of all ages and can be quite efficient for certain types of problems. The technique of using repetitious cycles of successive approximation has received a lot of bad press over the past century or two, generally perpetrated be over-zealous mathematicians seeking "beauty and rigor" in their chosen field. Much of what we know today, however, was motivated by such enthusiastic"messing about". Guesstimation can be thought of as a preliminary, informal experiment motivated by intuition.

Its most obvious advantage, simplicity, lends it well to use by young children. What it gains in common sense application, however, it loses in efficiency. Generally, trial and error methods consume considerable time and effort yet may provide the only initial entry into a novel situation. It is interesting to note that modern computers utilize adaptations of such techniques in order to compute square roots, trigonometric functions and a host of other fancy tricks.

For example:

Suppose you have enough money to build 100 meters of fence to protect your new garden. If you want the largest possible garden, what would be the dimensions of the fence enclosure?

First we list the given facts, conditions and goal:

Facts - 100 meters of fence.

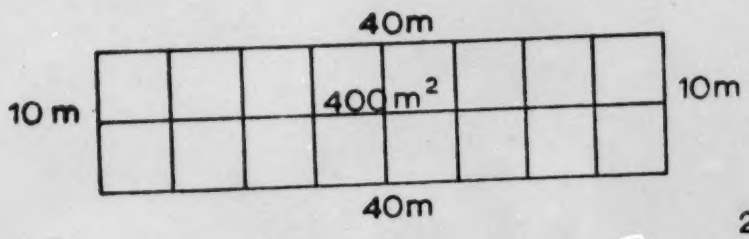
Conditions - Garden must be largest possible area,

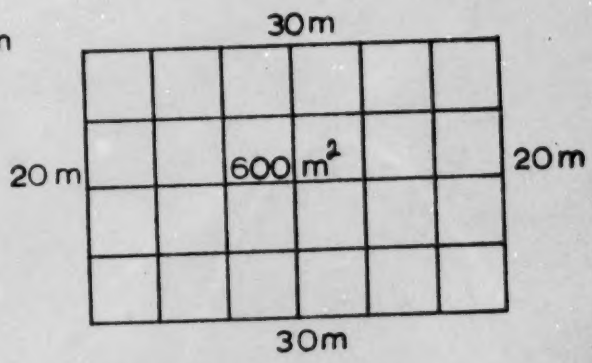
- No breaks in fence.

- Edges of enclosure must add up to 100 meters.

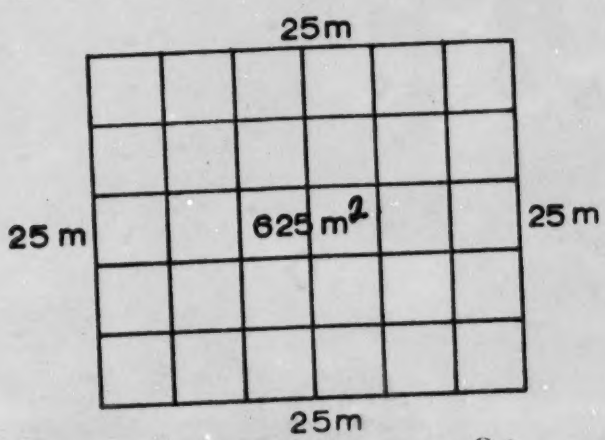
Goal - Find dimensions of enclosure

Let's try a few initial guesses to help us understand the problem and perhaps discover a pattern. A piece of graph paper will help keep things organized.





It seems that as the enclosure gets "fatter," more area is available for the garden. Let's try the fattest four-sided figure.



If we are satisfied with a four-sided figure (implicit condition) a square enclosure seems to afford the most generous garden. If more area is required, enclosures with more or less than four sides might be explored. It is interesting to note that a circle affords the largest garden for any fixed boundary (if you can plow in circles). How much extra planting room does a circular enclosure provide over the square garden?

Working Backwards: One method often used when the solution involves describing the steps which must be taken in order to achieve a stated goal is working backwards. When using this strategy one starts with the end result and determines a sequence of necessary preceeding conditions (analysis) which, when carried out in reverse (synthesis), will generate the desired result. To be useful, the analysis must proceed towards a known state--something already known. The process is useful for various geometric and practical problems. It may provide an initial wedge into an otherwise opaque problem situation. For example:

> A pub normally sells beer in 4-pint and 9-pint pitchers (the owner refused to convert to metric). An old-time patron was too thirsty for the small pitcher but not thirsty enough for a large one. He asked the bartender to draw him exactly 6 pints. The bartender thought for a few minutes and said, "I can fill your order using only one 4-pint and one 9-pint pitcher, but will have to wait until a couple more customers arrive." How did he do it?

To solve this problem, we again list the given facts, conditions and goal:

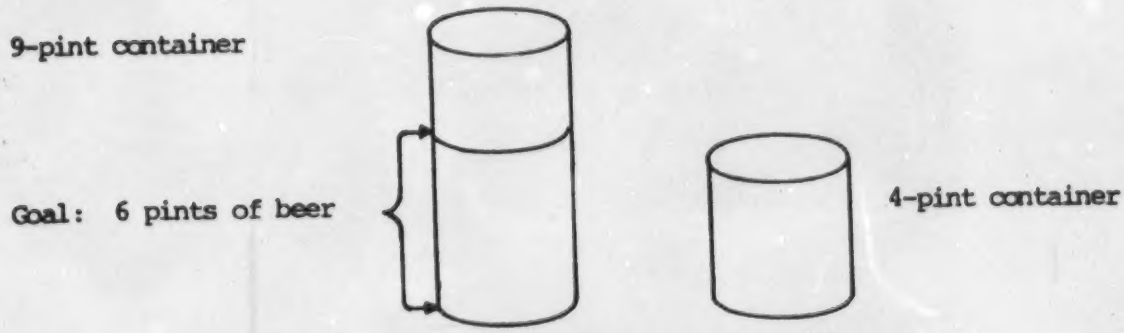
Facts - We have a 4-pint and a 9-pint pitcher

Condition - We have a large quantity of beer available

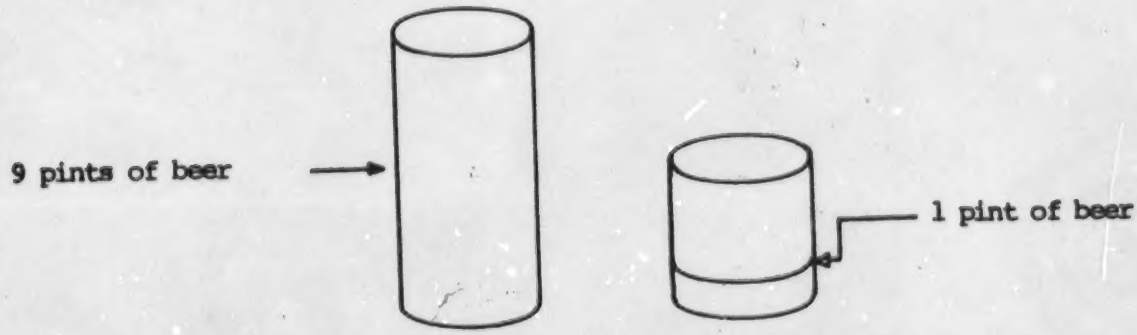
- We don't want to throw away any beer

Goal - Put 6 pints of beer in the 9-pint pitcher.

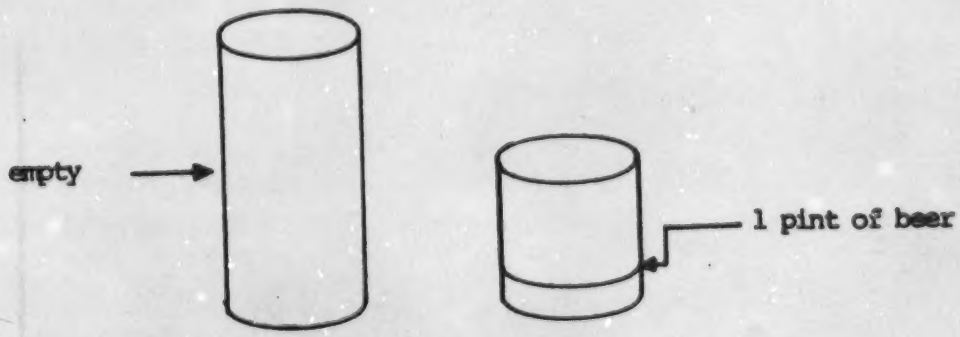
Working backwards is a convenient strategy for this problem. First we temporarily assume the problem is solved and imagine we have the 6 pints of beer. We can then work backwards looking for necessary preconditions until we arrive at a known condition. We might even try using water in order to conduct an actual experiment.



A sufficient precondition for the goal is to have 1 pint in the small container and the large container full since we could pour off 3 pints by filling the rest of the 4-pint pitcher, leaving 6 pints.

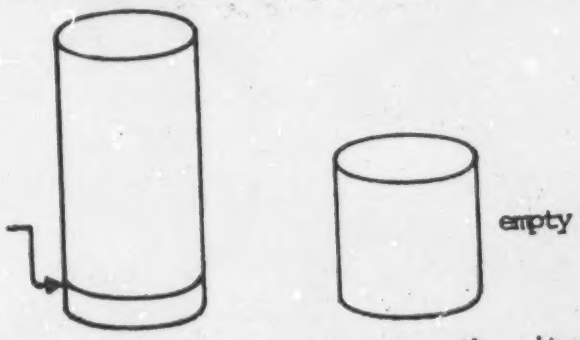


This situation can be easily reached from the situation below.



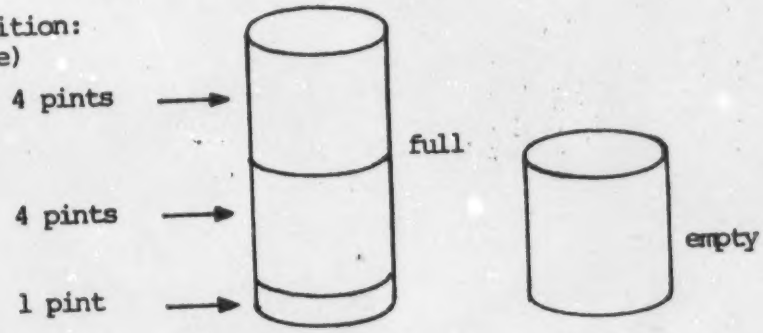
The situation below can then be accomplished.

3rd precondition:



Finally, the 3rd precondition can be reached from the situation below if we realize that we can pour off two 4-pint pitchers leaving one pint in the large container. To avoid wasting any beer a couple of thirsty customers will have to drink these two pitchers before reversing the steps to reach the goal.

4th precondition:
(known state)



Solution: After two customers arrived the bartender filled up the large pitcher, poured off one 4-pint pitcher and served it to a waiting patron. After she finished, the bartender poured off another 4 pints and served it to the other thirsty patron. When the pitcher returned, the remaining pint was poured into it. The large container was then refilled. The small container was subsequently filled by pouring off 3 pints of beer from the 9-pint container, leaving 6 pints. It makes you thirsty just thinking about the plight of the poor bartender.

Reduce to a Simpler Case: Insight can be gained into many problems by assigning simpler values or reducing the level of complexity specified in the problem. Many times by simply removing the cognitive stress induced by large numbers, fractions or decimals,
children are able to concentrate on the interrelationship between problem elements. Once this simple version is solved, the original values can be substituted and solved in a similar manner. For example:

> Suppose you're throwing a party and are planning to serve canned shrimp. Three different size cans are available. Which gives you the best buy?
>
> | | net weight | |
> |---|---|---|
> | 1. | 227 grams | @ $3.45 |
> | 2. | 341 grams | @ $4.78 |
> | 3. | 454 grams | @ $6.81 |

What are the given facts, conditions and goal?

Facts - Three different size cans with different prices.

Conditions - Must buy whole cans.

Goal - Buy can offering best buy.

To develop a rule for finding the best buy- let's use a "model" can with "nice" numbers.

200 grams net @ $4.00

It is somewhat easier to determine the price per gram since the answer comes out even.

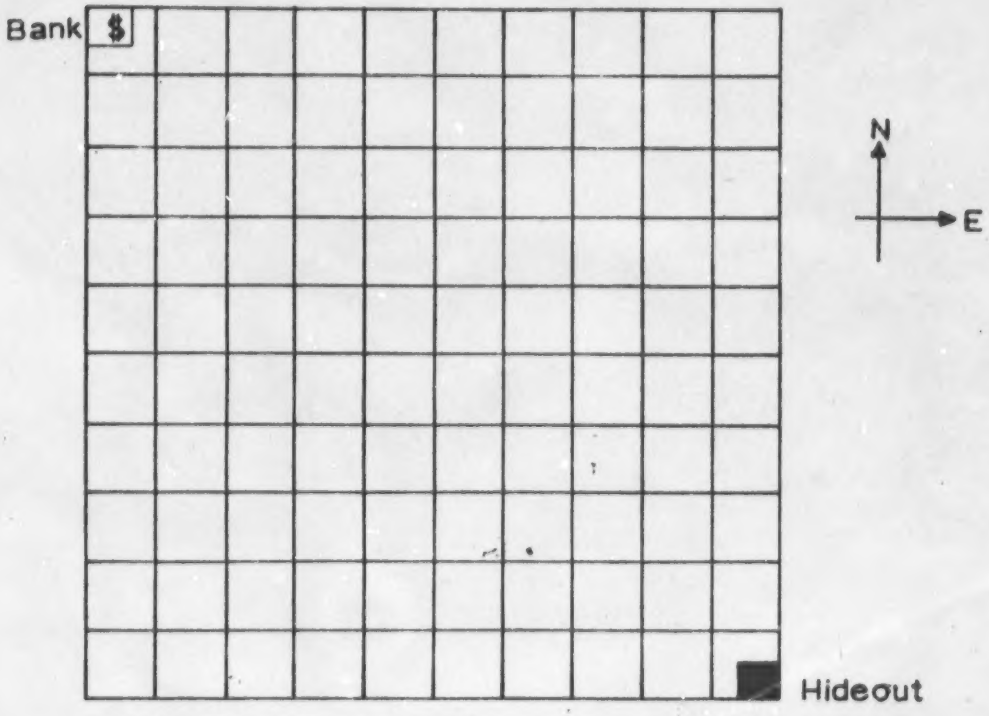
400 ÷ 200 = 2¢ per gram

Using a calculator we can follow the same rule with the actual values and discover that can is the best buy.

1. 345 ÷ 227 = 1.5¢ per gram
2. 478 ÷ 341 = 1.4¢ per gram
3. 681 ÷ 454 = 1.5¢ per gram

An additional example where simplifying helps with the solution is described below.

A bank robber was planning his getaway route from the bank to his hideout. He wanted to figure out how many routes were possible and if one was shorter than the rest. He drew the map below to help with the problem.



The robber has to drive on the streets of the hundred block section of town and wants to get to the hideout as quickly as possible (always drive East or South). How many routes are there and which one is shortest?

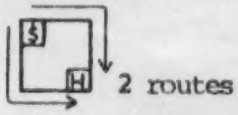
To solve this problem let's first list the given facts, conditions and goal.

Facts - 10 x 10 block section of the city
- blocks are square

Conditions - Must drive on streets
- Must drive East or South

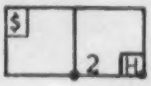
Goals - Find total number of routes
- Find shortest route

Since the number of routes for a 10 X 10 section is quite large, it might be convenient to look at smaller sections and see if we can discover a pattern. In a 1 X 1 section we find two possible routes.



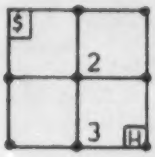
2 routes

For a 1 X 2 section we find 3 possible routes.



3 routes

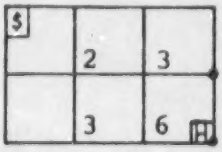
For a 2 X 2 section we find 6 possible routes. Can you find them?



3 routes

6 routes

For a 2 X 3 section we find 10 routes.



4 routes

10 routes

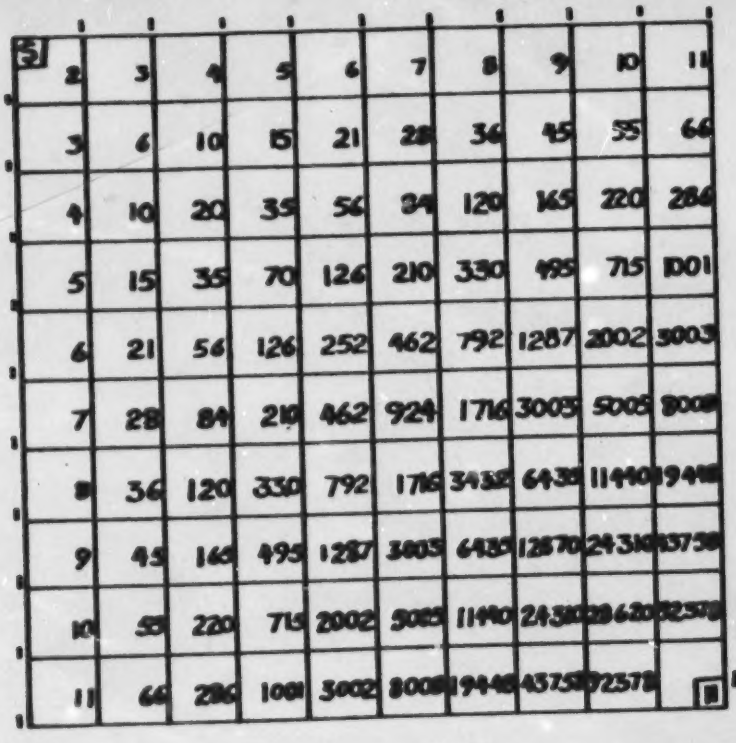
And for a 3 X 3 section we find 20 routes.

4

10

20 routes

Observing the patterns in the figure above (along with a bit of faith) we can easily complete the problem for a 10 X 10 section. Notice that in order to find the number of possible routes at any corner one needs only to add the number of routes to the corners immediately to the North and to the West. (Note that there is only one route to any corner around the outer edge.)

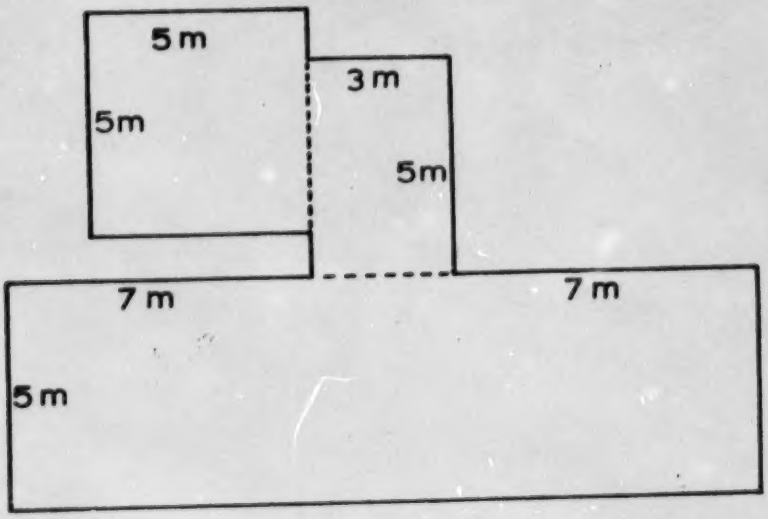


The robber has 184, 756 possible routes to choose from. Without simplifying the problem and subsequently looking for patterns, the solution would be time-consuming indeed. How long would it take to "check out" all the routes if only 20 could be driven in one day? It is interesting to note that all the routes are the same length. Can you prove it?

Another frequently used simplification strategy involves breaking a complex problem into smaller, more palatable chunks. Each of the sub-problems can then be solved and, when taken together, comprise a solution to the original problem. Just as a builder must concentrate first on the grading, then the foundation, framing, roofing and other elements of constructing a new house, many mathematical problems succumb to a similar subcontracting of effort.

Examples of problems which are susceptible to chunking abound in the fields of measurement and geometry. For example:

A carpet layer must calculate floor area prior to determining the total price for a job. How many square meters of carpet will be required for the floor plan below?



It is useful to break the floor plan into rectangles. Add the results to determine the total floor space.

$$5\text{ m} \times 5\text{ m} = 25\text{ m}^2$$

$$3\text{ m} \times 5\text{ m} = 15\text{ m}^2$$

$$5\text{ m} \times 17\text{ m} = \underline{85\text{m}^2}$$

Total floor area 125 m^2

Conduct an Experiment. Though most real world problem solving (business, science, engineering) relies heavily on experimentation, elementary aged children are rarely encouraged to conduct experiments as a practical problem solving strategy. Classical experiments (scientific method) are remarkably similar to the four-step general problem solving plan. First, one must understand the problem and define a goal (hypothesis). Next a strategy must be formulated (design an experiment) and carried out (data collection). Finally, we evaluate the results of the experiment(s) and test the hypothesis for accuracy.

In mathematical problem solving experiments generally consist of carrying out some systematic procedure using a physical material or some sort of graphic representation. Sketches, graphs and tables are useful when organizing data gathered from an experiment. Once the data has been organized, searching for patterns or general rules (functions) generally aids the solution process.

The following list of suggestions is helpful in trying to solve a problem by experimentation. Generally, a coordinated effort using two or more aids is necessary to achieve success.

Aids to conducting an experiment:

1. Design a physical model
2. Make a sketch or diagram
3. Collect data and organize it in a table
4. Construct a graph
5. Search for a pattern
6. Develop a general rule (function)

For example:

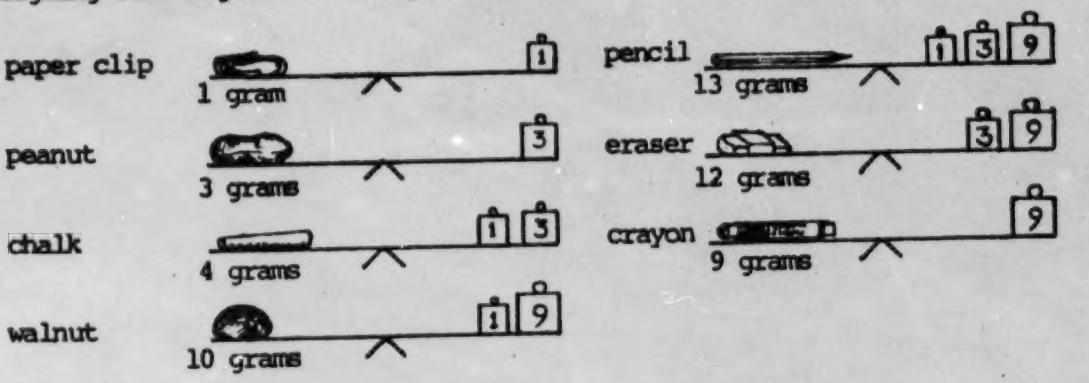
How many different masses can be weighed on a pan balance using only a 1 gram, 3 gram and 9 gram mass piece? Sounds simple enough. What are the given facts, conditions and goal?

<u>Facts</u>
- We can use a pan balance
- We have 3 mass pieces (1g, 3g and 9g)

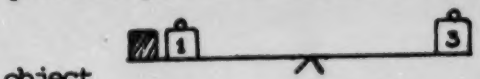
<u>Conditions</u>
- We can use only three mass pieces
- We can place the mass pieces both sides of the pan balance.
- We can only weigh objects to the nearest whole gram.

<u>Goal</u>
- How many different masses can we weigh?

Conducting an experiment will help up with the solution. First, let's try weighing some objects with a pan balance.



The answer seems to be 7 (not including 0 grams), yet if we realize that the conditions allow us to put mass pieces on either or both pans, additional masses can be weighed.



object
(2 grams)

Let's fill in a table in order to list all possible combinations.

Mass weighed (g)	Mass piece on left pan	Mass piece on right pan
1		1 g
2	1 g	3 g
3		3 g
4		1 g & 3 g
5	1 g & 3 g	9 g
6	3 g	9 g
7	3 g	1 g & 9g
8	1 g	9 g
9		9 g
10		1 g & 9 g
11	1 g	3 g & 9 g
12		3 g & 9 g
13		1 g & 3 g & 9 g

All whole number masses from 1 to 13 can be weighed! Try including one 27 gram mass piece along with the original three pieces and determine the number of different masses which can be weighed. What is the minimum number of mass pieces necessary to weigh every whole number mass from 1 to 1000 grams? Can you think of a situation where such a set of mass pieces would be extremely convenient?

Add Elements to the Problem Situation. One final strategy which is rarely used but worthwhile mentioning is the addition of new elements to the problem to ease solution. A few problems are virtually insoluable without applying this technique. A classic example is the following:

> Suppose you are given a bag full of marbles and are told that all the marbles are identical. You are allowed to take out only one marble at a time and can look at it though it must be returned to the bag. Under the stated conditions, how can we reasonably estimate the number of marbles in the bag?

If we remain limited to the given elements (bag, marbles, see one at a time and return), no solution short of guessing is available. If, however, we break out of our traditional mind set and consider adding a new element to the situation, a solution becomes possible.

> Solution: First remove one marble and carefully observe its characteristics. Find a supply of identical marbles (say, 20) and mark each with an (s). Return the original marble and the 20 new "seed" marbles to the bag and shake the bag until the "seeds" are thoroughly mixed with the original marbles. Next, conduct an experiment by removing one marble, noting if it is a "seed" or original marble, return it to the bag and shake it thoroughly. Organize the data in a table as below.

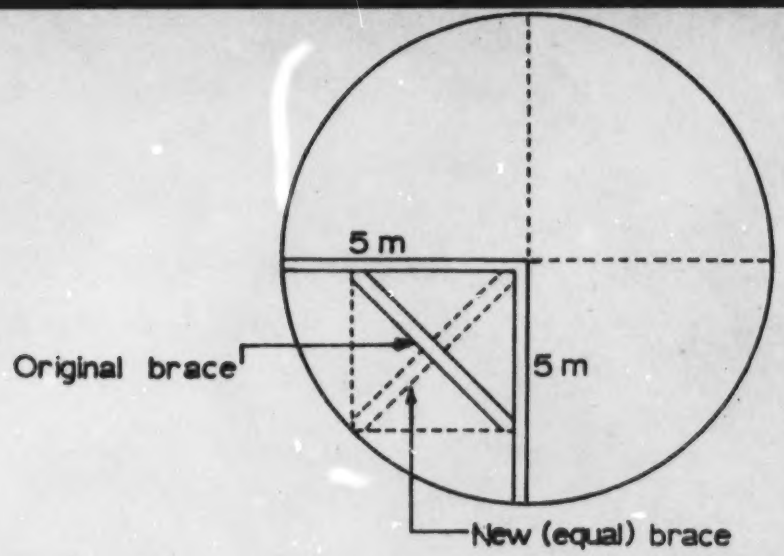
Seed Marble	Original Marble

After sufficient trials, the laws of probability inform us that for every 20 seed marbles (the total number seeded) we draw, we will have also drawn out the number or original marbles in the bag! For example if in 100 trials we drew out 40 seed marbles and 60 original marbles, we predict that there were 30 marbles in the bag originally. Remember, for each set of 20 seed marbles drawn (2 X 20 = 40) we should have drawn out the number of original marbles in the bag (2 X 30 = 60).

The word "sufficient" is very important here since a sample of 10 or 20 marbles is not large enough to allow the laws of probability to be predictive. Samples of 100 to 500 should do the trick of the bag is shaken well. It is interesting to note that there is no guarantee of drawing every seed or original marble no matter how large the sample. The proportion of seed to original marbles is really what we're looking for. It is even possible (though very remote) that the same seed and original marble are repeatedly drawn throughout the experiment!

This process of seeding is used in medical diagnosis of various blood diseases. A radioactive material which is absorbed by healthy and diseased cells in differing proportions is injected into the blood stream. By measuring the ratio between the radioactivity of healthy and diseased cells, an accurate count of diseased cells can be established.

The previous problem in submarine geometry further illuminates the usefulness of this technique. By inserting the opposite diagonal in the diagram, the solution becomes trivial; since both braces are equal and the new one is a radius of the circle (5 meters), the original brace must also be 5 meters long.



Step 3:Carrying Out The Strategy

One the problem situation is understood (facts, conditions and goal) and the difficult task of devising a solution strategy has been completed, the next st p is to get your feet wet and apply the chosen strategy. It should be pointed out that in practice steps 1, 2 and 3 are not carried out in isolation from each other but in fact may be far more interdependant. In trying to understand a problem it may be useful to immediately apply a favorite strategy and gather a bit of data which may, in turn, further understanding and lead to an effective solution strategy. Initially, however, it is recommended that novices proceed through the steps as described, yet be cautioned against getting stuck at any one level.

Carrying out a solution strategy primarily requires persistance. The task is one of assessing if the chosen strategy generates meaningful clues towards unraveling the problem. These clues may take the form of patterns or general rules relating the problem to a known example.

Throughout the process of working through a strategy it is important to:

1. Keep accurate records (tables, sketches, etc.)
2. Stick to the chosen strategy until some evidence suggests specific changes.
3. Carefully monitor each step in the solution for accuracy and logic

4. If after a reasonable period of time no headway is made, put the problem aside for a day or two then try it again or look for another strategy.

Step 4: Evaluating The Results

Looking back over the solution of a problem is beneficial for two reasons. First, it provides an opportunity to evaluate and refine the results. Second it brings the process of solution into clear focus. At no time are children in a better position to clearly understand and explain the somewhat intuitive processes in which they are engaged than immediately following a successful problem solving experience. Not only does the child evaluate the validity of the results, but more importantly, coordinates the current solution with previously solved problems.

Nothing causes problems like solutions; the real benefit to the learner extends much further than the immediate result of any particular problem. If each solution is carefully reconsidered in light of past experience and co-ordinated with prior results, the ability to solve novel problems in the future may be enhanced. If the goal is to improve general problem solving abilities, children must be strongly encouraged to reexamine their successes, refine their results and imagine problems which have been or could be solved with similar strategies. Otherwise each problem is but a lonely star shining in the unknown.

Summary: General Problem Solving Plan

Step I. Understand the problem.

A. Know the given facts, conditions and goal.

Step II. Design a solution strategy.

A. Guesstimation (trial and error).

B. Work backwards

C. Reduce to simpler case.

D. Conduct an experiment.

1. Design a physical model.
2. Make a sketch or diagram.
3. Collect data and organize it in a table.
4. Construct a graph.
5. Search for a pattern.
6. Develop a general rule (function).

E. Add elements to the problem situation.

Step III. Carry out the strategy.

A. Persistently follow through with the solution strategy.

B. Maintain accurate record of the data collected, etc.

Step IV. Evaluate the results.

A. If a solution is uncovered, refine the results and try to relate them to other problems.

B. If not, re-evaluate your understanding and seek a new solution strategy.

CLASSROOM ACTIVITIES

- Whole-Class Problem Solving Warm-Ups
- Three Problems With Solutions
- Bibliography

Whole Class Problem Solving Warm-Ups

Breaking Into Groups-Of-Four

Present the problem of dividing the class into groups-of-four with each student having an equal chance of being in any group. In a teacher directed large group discussion, have the class brainstorm possible random selection techniques and discuss the effects of each suggestion. Start by listing all the suggestions on the board without comment.

Then each suggestion should be carefully discussed and the most effective methods chosen.

Some possible suggestions for a class of 32 include:

1. Deal out an appropriate sized, shuffled deck of cards and form groups of people with the same number.
2. Alphabitize the class list and count off by fours.
3. Draw names out of a hat four at a time.
4. Have students draw out of a hat containing four sets of cards numbered one through eight. Those with the same number go together.

Brainstorming

Using questions similiar to those below, have the whole class work together to "brainstorm" possible answers. Write responses on a piece of poster board or on the blackboard. Discuss each response in detail making sure everyone has an opporunity for input.

1. Name 10 things that come in millions.
2. Name 10 things that you can do with a blade of grass.
3. If you had to leave home quickly and could only take one suitcase with you, what would you put in it?
4. This school would be a better place if everyone .....
5. Name 10 things you can do with a pencil sharpener.
6. List 10 words that describe this class.
7. List 10 things you can do now that you couldn't do last year.

Mind Reading

Ask each student in your class to think of a number between 0 and 50. Both digits must be odd and they cannot be alike. For example, you can not use 11. Have them secretly write their number on a piece of paper.

Before starting the activity, secretly write the number 37 on the board and cover it with a piece of paper. After the whole class has written down their guesses, ham it up a bit, asking them to concentrate on their number so you can read their minds. Dramatically remove the paper covering the number on the board and ask how many students chose the # 37. A surprising number of students will probably choose this number.

Try the same activity again only ask for a number between 50 and 100 made up of two different even digits. This time write the digits 6 and 8 on two large cards (or use playing cards). Again ask them concentrate then show the number 68. Many will have chosen this number. Reverse the cards (86) and explain that they may have gotten their wires crossed choosing this number.

Though this seems like a magic act, in fact there are only eight numbers that fit the first category and for some curious psychological reason, most people choose 37 (the next most popular number is 35). The second situation is restricted to eight choices as well with 68 being the most popular. Of course reversing the cards shows 86 thereby increasing your mind reading success.

Race to 21

Race to 21 is played by two people taking turns counting one or two numbers until someone reaches 21. Choose who is to start. The first person can either say "1" or "1,2". The other contestant counts on from where the first person left off again saying one or two numbers. This process of alternate counting continues until 21 is reached. The winner is the person saying 21.

This game can be played with all ages and various sized groups. The teacher should play against volunteers in the class to help students learn the counting procedure. Then have the children pair off and see if they can develop a winning strategy.

A clever way to unravel this problem is to play the game backwards. If you want to say 21, which previous number must you say? If you say 18, your opponent could count "19", or "19, 20". Either way, you could claim 21. If you must say 18, which previous number must you say? This time 15 is the key number since saying it guarantees you can claim 18. Continue this process.
If you say 3 you are guaranteed all the key numbers to 21. To say 3 your opponent must start (unless you are very lucky). However if you have been chosen to start, try to capture a key number as soon as possible. Once you say a multiple of 3, you are guaranteed to be a winner (if you don't make any mistakes).

Once your students have mastered this version, try counting one, two or <u>three</u> numbers at a time. Another interesting twist is to make 21 "poison" - the person who says 21 loses.

Pico-Fermi-Nada

The goal of "Pico-Fermi-Nada" is to guess a secret three digit number. The leader writes any three digit number on a piece of paper and hides it. Individuals or teams take turns guessing three digit numbers. Each time the leader gives one of the clues below:

1. Pico - At least one digit is correct and in the correct position.
2. Fermi - At least one digit is correct but in the wrong position.
3. Nada - No digits are correct.

Try to guess the secret number in the fewest possible attempts. Teams can play against each other, alternating guesses and learning the clues from each other. The whole class can also play as a unit trying to reduce the number of guesses from one trial to the next.

Here is a sample game: Secret number (714).

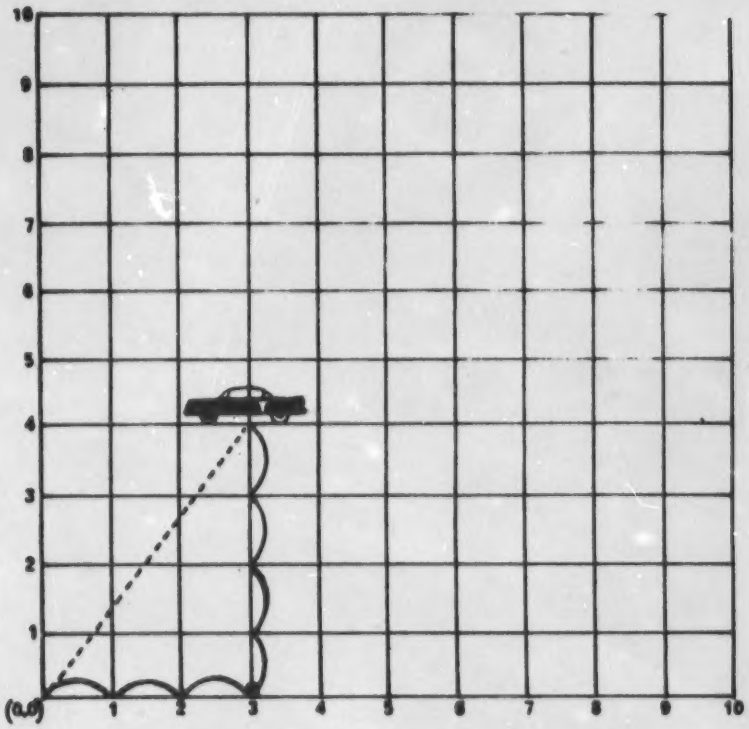
	Guess	Clue - Comments
1	987	Fermi - At least one digit correct but in wrong position.
2	999	Nada - No digit correct - 9 not digit in secret #.
3	888	Nada - No digit correct - 8 not digit in secret #.
4	978	Fermi - 7 correct digit - must be in 1st position.

	Guess	Clue - Comments
5	654	**Pico** - At least one digit correct - 6 not digit in secret #.
6	235	Nada - 2, 3, and 5 not digits in secret #, 4 in 3rd position.
7	714	Correct Guess - Could have guessed 704 requiring one additional clue.

The comments in the table above provide examples of logical deductions which can be made after each guess and clue. With experience, guesses can be carefully chosen to provide as much information as possible from each of the clues. With practice any three digit number can be "guessed" in 7 to 10 tries.

Car Race

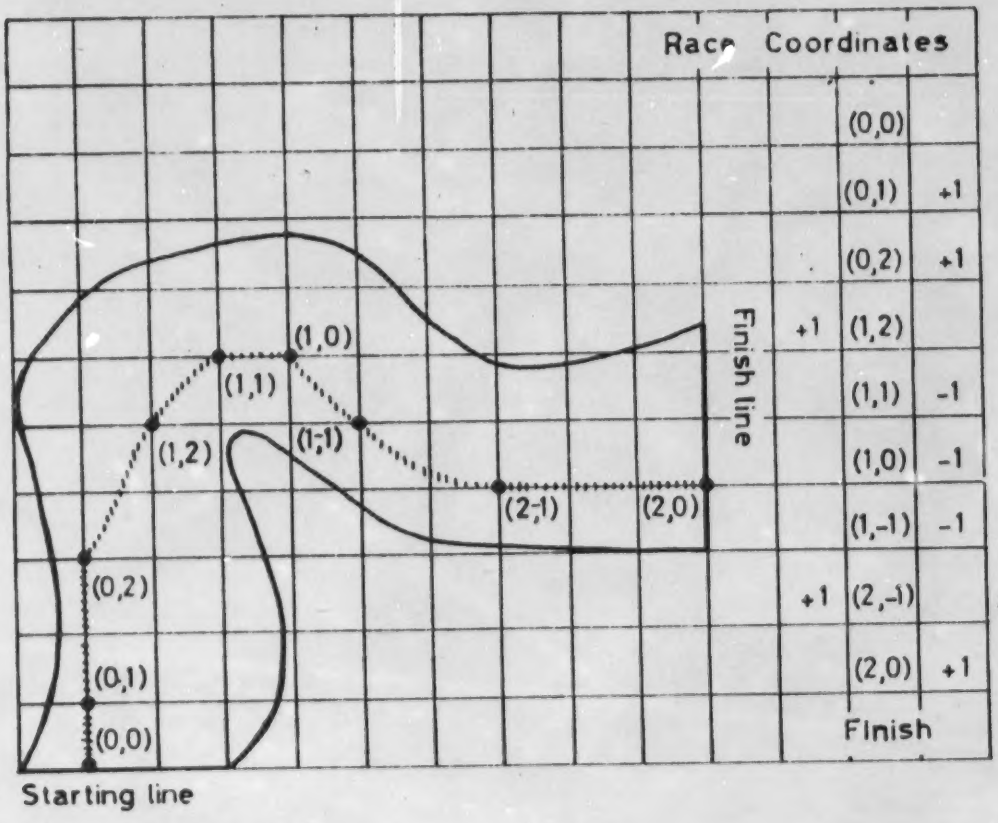
Before playing Car Race, show your class how to graph coordinate pairs. Beginning at (0,0) on the coordinate graph below, explain that the first number in the pair (3, 4) tells you to go right 3 spaces and the second tells you to go up 4 spaces. The car shows you where to end up.



To play Car Race draw a wide road on a piece of 1 cm squared paper. Each player begins at the starting line They each have their own (0,0) starting coordinate. To race, each player in turn can add (or subtract) 1 from their first or second coordinate and move their car to the spot indicated by the new coordinate pair. Notice each person starts at (0,0) so the only two first moves are (0,1) or (1,0) for all the players. (Avoid negative numbers until the class decides they need them later in the problem. See example below). After everyone has moved once, again add (or subtract) 1 to either coordinate and move on from the last position. Continue adding (or subtracting) 1 to each subsequent move and try to negotiate the curves without crashing into the edge of the road or another car. The first one to the finish line

without crashing is the winner.

Here is a sample game:



Begin by having the whole class race one car. Later, break onto groups of 2 or 3 and have the students work on efficient strategies by racing against each other.

PROBLEM STARTER

Name: _____

What's The Difference?

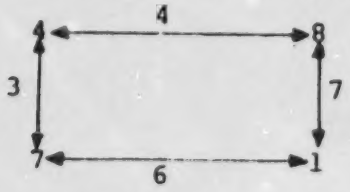
Pick any four numbers and write them in a square as below.

4 8

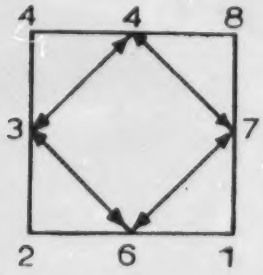
7 1

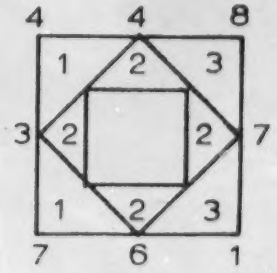
Next connect any two numbers without going across the middle. Find the difference and write it at the centre of the line.

Continue with the other pairs.



Now connect these numbers with lines and find their differences.





This time all the answers are the same.
Will this always happen for any four starting numbers?
Will you always end up with 2 for a final answer?
What happens if you start with bigger numbers?
This problem needed 3 steps to get to a common difference.
Can you find 4 numbers that needs 4 steps? 5 steps? 6 steps?

Hint: Try many combinations of 4 numbers. Keep a a record of the orignal 4 numbers, the number of differences and the final answer.

SOLUTION

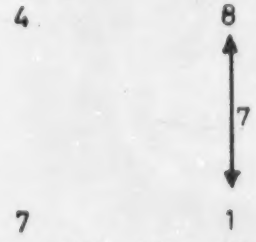
What's The Difference?

Pick any four numbers and write them in a square as below.

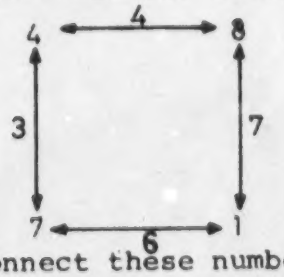
4 8

7 1

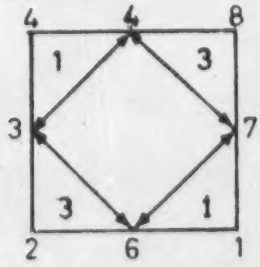
Next connect any two numbers without going across the middle. Find the difference and write it at the centre of the line.

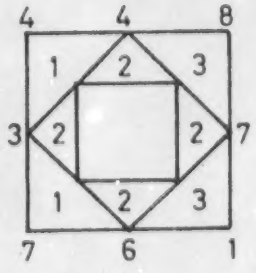


Continue with the other pairs.



Now connect these numbers with lines and find their differences.





This time all the answers are the same.

Will this always happen for any four starting numbers?

Will you always end up with 2 for a final answer?

What happens if you start with bigger numbers?

This problem needed 4 steps to get to a common difference.

Can you find 4 numbers that needs 4 steps? 5 steps?

6 steps?

Solution: What's The Difference?

Understand the Problem

Facts - Four numbers are placed on the corners of a square.

- Differences are found for each pair of adjacent numbers.
- These four numbers are again connected in a square array and difference found as before. The process is continued until you arrive at a common answer.

Conditions - Always compute the positive difference between two numbers. For example the difference between 3 and 5 is 2 not -2.

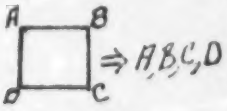
Goal - Find out if you always end at a common difference, for any four numbers. Does this final answer vary? Can you find four numbers that require 4 steps to arrive at a common difference ? 5 steps? 6 steps?

Design a Solution Strategy

Try many combinations of four numbers. Record the original numbers, the number of steps until you reach a common difference and the final answer.

Carry Out Plan

The table below lists the results of several experiments:

4 Numbers	Number of Steps	Common Difference
4, 8, 1, 7	3	2
3, 88, 99, 57	3	27
1, 2, 8, 4	5	2
1, 2, 4, 8	6	2
1, 3, 9, 27	5	8
12, 77, 2, 25	3	42
2, 5, 7, 1	4	2
3, 6, 9, 12	1	3
15, 8, 11, 4	2	4
		

Evaluate Results

Every example seems to eventually arrive at a common difference. The final answer varies and the number of steps ranges from one to six. We have been unable to find an example that requires more than six steps to arrive at a common difference. Perhaps your class can find a set of four numbers that requires more than 6 steps.

An interesting extension involves rearranging the order of a set of numbers and observing the required number of differences to arrive at a common answer. (See [1, 2, 8, 4] and [1, 2, 4, 8] above). The final difference seems to be the same but the number of steps varies. Is this the case for other sets of numbers? A calculator may help with this one.

PROBLEM STARTER

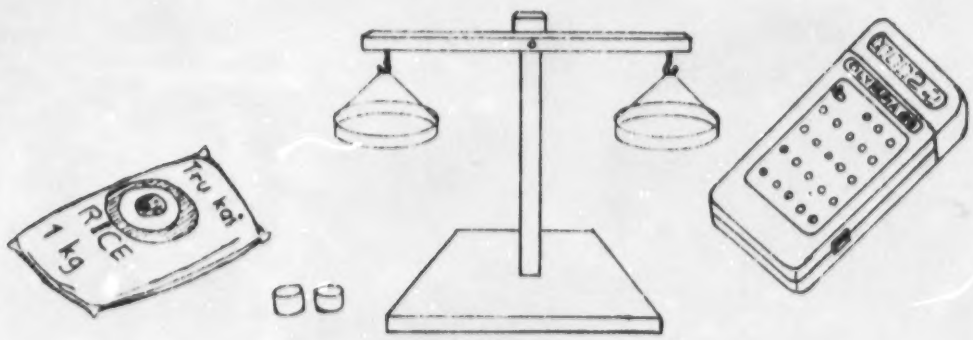
Name: _____

Grains Of Rice

How many grains of rice are in a bag of rice?

100 grains? 1000 grains? 1,000,000 grains?

To find out you need:



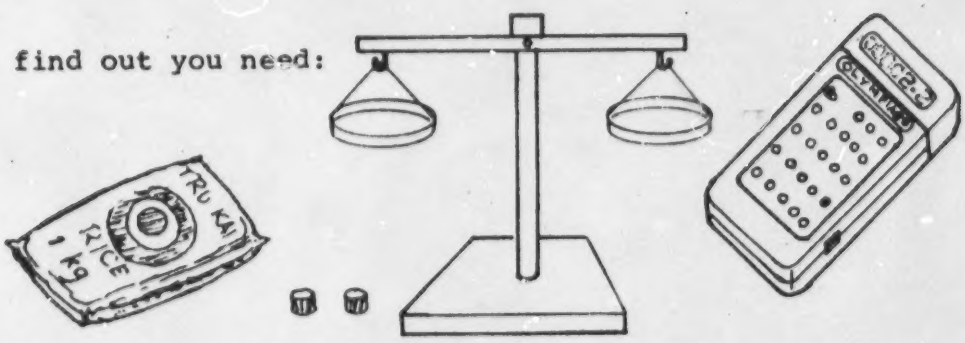
> Hint: Measure out one small cup full of rice. Count the grains in one cup. Use the pan balance and calculator to help find the total number.

SOLUTION

Grains of Rice

How many grains of rice are in a bag of rice? 100 grains? 1000 grains? 1,000, 000 grains?

To find out you need:



Solution: Grains Of Rice

Understand the Problem

Facts - A bag of rice contains many grains.
- Grains are about the same size.
- Two common sizes are 1 kilogram and 1 pound bags.

Conditions - Have access to only pan balance, small cups and calculator.

Goal - Find the approximate number of grains in a bag of rice.

Design a Solution Strategy

Measure out one small cup of rice. Any small coffee-creamer size cups will work. Count the grains. For younger children, a team approach to this counting task may be in order. Using the balance, weigh out quantities of rice one cup at a time to find the total

number of cups in a bag. This process can be speeded up considerably by first measuring out one cup on each side of the balance. Next, place both measured cups on the same pan and pour rice on the other pan until it balances. Again put the two piles together on one pan (now four cups) and weigh out an equivalent amount on the balance. Continuing this process of doubling (1, 2, 8, 16 ... cups) one is able to quickly determine the number of cups in the bag. Using a calculator, multiply the number of grains in one cup by the total number of cups to find the total number of grains in the bag.

Carry Out Plan

The number of grains in a coffee-creamer cup is about 1135. A kilogram bag of rice contains approximately 38 creamer-cups of rice. The total number of grains is 38 x 1135 = 43,130 grains.

Evaluate Results

The answer above is only approximate. It's accuracy depends on the consistency in grain size and accuracy in measurement. Obviously there will be errors, however, the answer does give a reasonable approximation to the number of grains.

The size of the measuring cup isn't important though a small one simplifies the counting process and minimizes error due to a fractional part of a cup left over when weighing out the whole bag.

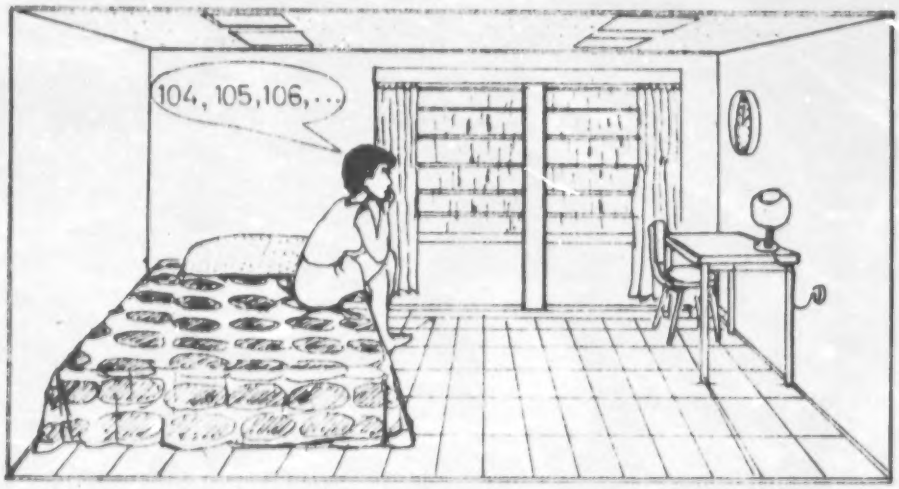
An interesting extention to this problem for older children is to consider how large a packing crate would be required to contain 1 million grains of rice. You might even want to set your class to work finding the number of grains of rice it would take to fill your classroom. Would a billion grains do it? How many grains of rice in one serving? How many people would it take to eat a room full of rice?

PROBLEM STARTER

Name: _____

Count On

On rainy afternoon Tara couldn't find anyone to play with so she decided to count to a billion. Can you find out about how long it would take Tara to finish? One hour? Two hours? All day?



How old are you in seconds?

Hint: Figure it takes about one second to say one number and use your calculator to help you crunch the big numbers.

SOLUTION

Count On

On rainy afternoon Tara couldn't find anyone to play with so she decided to count to a billion. Can you find out about how long it would take Tara to finish? One hour? Two hours? All day? How old are you in seconds?

Solution: Count On

Understanding the Problem

Facts - Use natural numbers (1, 2, 3, 4, 5, ...)

Conditions - Count steady at a rate of one number a second

Goal - How long will it take Tara to count to one billion?

Design a Solution Strategy

For simplicity, let's estimate that it takes one second to count each number. Using a calculator, figure out how high Tara could count in a hour; a day; a week; a year. Organize the results in a table.

Carry Out Plan

The following table lists how high you could count in a second, minute, hour, day, etc, if you count one number each second.

Time	Count
1 second	1
1 minute	60
1 hour	3600
1 day	86,400
1 week	604,800
1 year	31,449,600
10 years	314, 496, 000 (most calculators "give up" here and overflows.)
30 years	943, 488, 000 (almost there)
32 years	1,006,387,200 (we made it!)

Evaluate Results

This problem clearly demonstrates the limitations of the simple hand-held calculator. The largest number most calculators can show accurately is 99,999,999, far smaller than our goal. In fact we can calculate only a little more than 3 years worth of counting. A bit of "old fashioned" arithmetic quickly shows that Tara would have to count day and night for over 31 years to reach her goal! It would certainly be a long rainy afternoon.

Using the table it is easy to figure your age in seconds. Anyone older than three years will have to figure it out by hand, however, unless they have a very fancy calculator. As of 3 October 1980 the author was exactly 1,037,836,800 seconds old. Can you figure out his birthday and age?

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